Journal of Statistical Physics, Vol. 111, Nos. 3/4, May 2003 (© 2003)

Higher-Order Spectral Densities of Fractional Random Fields

V. V. Anh,[1] N. N. Leonenko,[2] and L. M. Sakhno[3]

Received November 30, 2002; accepted October 7, 2002

This paper presents the second- and higher-order spectral densities of stationary (in space) random fields arising as approximations of rescaled solutions of the heat and fractional heat equations with singular initial conditions. The development is based on the diagram formalism and the Riesz composition formula. Our results are the first step to full parametrization of higher-order spectra of some classes of fractional random fields.

KEY WORDS: Fractional kinetic equation; fractional diffusion equation; higher-order spectra; long-range dependence; non-Gaussian scenario; Mittag–Leffler function.

1. INTRODUCTION

Partial differential equations (PDE) such as the heat or fractional heat equation have been used to represent many natural processes (see Schneider and Wyss,[37] Kochubei,[26] Mainardi,[28] Podlubny,[32] Mainardi and Gorenflo,[29] Mainardi *et al.*[30]). On the other hand, it is well documented that non-Gaussian random fields with long-range dependence (LRD) have been useful in describing data arising in many areas such as turbulence, finance, porous media and anomalous diffusion (see Barndorff-Nielsen and Shephard,[9] Hilfer,[21] Metzler and Klafter[31] and the references therein). Those random fields, which are solutions of PDEs with random initial

[1] School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane QLD 4001, Australia; e-mail: v.anh@qut.edu.au

[2] School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4YH, United Kingdom; e-mail: leonenkon@Cardiff.ac.uk

[3] Department of Mathematics, Kyiv University (National), Kyiv 01033, Ukraine; e-mail: lms@mechmat.univ.kiev.ua

conditions to describe non-Gaussianity, LRD and/or intermittency, have been a subject of extensive research in the current literature.

An introduction of rigorous probabilistic tools into the subject can be traced back to Kampé de Feriet[25] and Rosenblatt[35] who considered the heat equation with stationary initial conditions. Some mathematical aspects of the random initial value problems for the heat or fractional heat equation have been treated in Leonenko and Woyczynski,[27] Anh and Leonenko.[3–6] In particular, they have presented the non-Gaussian scenarios for the rescaled solutions of heat or fractional heat equations with strongly dependent initial conditions (see also their references). These limiting distributions (non-Gaussian, in general) are described in terms of multiple Wiener–Itô integrals. In a sense these results are analogous to the limit theorems for non-linear transforms of Gaussian processes and fields with LRD (see, for example, Taqqu,[41] Dobrushin and Major[17]), but the type of non-Gaussian limiting fields is different. In particular, these fields are stationary in space, while the random limiting processes and fields in Taqqu[41] and Dobrushin and Major[17] are non-stationary.

This paper presents a continuation of the above works of Leonenko and Woyczynski[27] and Anh and Leonenko.[3–6] We obtain the second- and higher-order spectral densities of stationary (in space) fractional random fields arising as approximations of rescaled solutions of the heat and fractional heat equations with singular initial conditions. The concept of higher-order spectra goes back to Brillinger,[11] Brillinger and Rosenblatt[13, 14] (see also Subba Rao,[38] Subba Rao and Gabr,[39] Rosenblatt,[36] Brillinger[12] and Priestly[33]). In these works, a non-parametric theory of estimation of higher-order spectra was considered for weakly-dependent discrete-parameter random fields. In this paper, we attempt to get full parametrisation of higher-order spectra of fractional random fields. This parametrisation is needed for identification of the parameters of these non-Gaussian random fields. In fact, the possible non-Gaussianity of a data set can be confirmed by using the bispectrum, which is zero for a Gaussian field but is non-zero for a non-Gaussian field (it should be noted that the usual second-order spectrum is the same for both cases). This important topic of non-Gaussian parameter identification, which requires some parametric forms of higher-order spectral densities, will be addressed in a subsequent paper.

These non-Gaussian models, particularly the fractional heat equation with different non-Gaussian random initial conditions outlined in Section 3 later, would be useful in modelling financial processes. In fact, these processes are known to exhibit scaling and have heavy-tailed marginal distributions (Barndorff-Nielsen and Prause,[8] Barndorff-Nielsen and Shephard,[9] Boyarchenko and Levendorskii[10]). The fractional Riesz–Bessel operator (extending the Laplacian) of the fractional heat equation (3.1) may be used to model the heavy tails via the Green function of this equation, while its random initial conditions may be tailored to reflect the global scaling behaviour of financial processes. Higher-order spectral densities will then play an essential role in the analysis of financial data.

Higher-order moments and spectra also appear in multifractal analyses of turbulence processes (Frisch[20]). Assuming that the corresponding multifractal formalism holds, the spectrum of singularities (i.e., the multifractal spectrum) of these processes can then be estimated from the L^p-spectrum of moments via the Legendre transform (Frisch,[20] Jaffard,[23] Riedi[34]). The multifractal formalism was discussed for multiplicative cascade processes and several other processes in Frisch.[20] A rigorous proof of this formalism was established in Jaffard[24] for some specific classes of functions in Sobolev and Besov spaces.

Higher-order spectral densities are also of interest in defining the long-range dependence and intermittency of non-Gaussian random fields (see Remarks 4–6 later). In fact, the singular behaviour of higher-order spectral densities at the origin and on the diagonals can be used to define higher-order LRD, and the corresponding result for the fractional heat equation allows to express the bifractal nature of the data in a more complete way than the corresponding second-order LRD and intermittency considered in Anh and Heyde.[2]

This paper is organized as follows. Section 2 presents the results for the random heat equation while Section 3 for the random fractional heat equation. Section 4 contains the proofs of some of these results. Appendix A contains the Riesz composition formula which plays a key role in our development. The diagram formula for the cumulants of multiple stochastic integrals and related definitions and notations are grouped together in Appendix B.

2. THE RANDOM HEAT EQUATION

In this section we present the second-order and higher-order spectral densities for the random fields arising as the limits of the rescaled solutions of the random heat equation. We consider the classical heat equation

$$\frac{\partial u}{\partial t} = \mu \Delta u, \qquad \mu > 0 \tag{2.1}$$

subject to the random initial conditions

$$u(0, x) = v(x), \qquad x \in \mathbb{R}^n, \tag{2.2}$$

where Δ is the Laplacian and $v(x)$ is a random field of the form $v(x) = h(\xi(x))$. The Gaussian random field $\xi(x), x \in \mathbb{R}^n$, and non-linear function $h(u), u \in \mathbb{R}^n$, are assumed to satisfy the conditions A, B, and C given below.

A. The field $\xi(x), x \in \mathbb{R}^n$ is a real measurable mean-square continuous homogeneous isotropic Gaussian random field with $E\xi(x) = 0$ and covariance function of the form

$$B(x) = (1 + \|x\|^2)^{-\varkappa/2}, \qquad x \in \mathbb{R}^n, \quad 0 < \varkappa < n.$$

B. The real function h is such that $Eh^2(\xi(0)) < \infty$.

The non-linear function h of condition B can be expanded in the series

$$h(u) = \sum_{k=1}^{\infty} \frac{C_k}{k!} H_k(u), \qquad C_k = \int_{\mathbb{R}^1} h(u)\, H_k(u)\, \varphi(u)\, du$$

of orthogonal Chebyshev–Hermite polynomials

$$H_k(u) = (-1)^k\, [\varphi(u)]^{-1} \frac{d^k}{du^k} \varphi(u), \qquad \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2},$$

$$u \in \mathbb{R}^1, \quad k = 0, 1, 2, \ldots$$

C. There exists an integer $m \geqslant 1$ such that

$$C_1 = \cdots = C_{m-1} = 0, \qquad C_m \neq 0.$$

The integer $m \geqslant 1$ is called the Hermitian rank of the function h (see, for example, Taqqu[41]).

In Anh and Leonenko,[3] the limit distributions of the rescaled solutions of the heat equation (2.1) with initial data (2.2) have been described in terms of their multiple stochastic integral representation. We recall this result in the following theorem.

Theorem 1. Let $u(t, x),\ t > 0,\ x \in \mathbb{R}^n$, be a solution of the initial value problem (2.1)–(2.2) with random initial condition of the form $v(x) = h(\xi(x))$, where a Gausian random field $\xi(x), x \in \mathbb{R}^n$, and a non-linear function $h(u), u \in \mathbb{R}$, satisfy the conditions A, B, C and $\varkappa \in (0, n/m)$, where m is the Hermitian rank of the function h. Then the finite-dimensional distributions of the random fields

$$X_\varepsilon(t, x) = \frac{1}{\varepsilon^{\varkappa m/4}} \left[u\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) - C_0 \right], \qquad t > 0, \quad x \in \mathbb{R}^n$$

converge weakly, as $\varepsilon \longrightarrow 0$, to the finite-dimensional distributions of the random fields

$$X_m(t, x) = \frac{C_m}{m!} [c(n, \varkappa)]^{m/2} \int_{\mathbb{R}^{nm}}' \frac{e^{i(x, \lambda_1 + \cdots + \lambda_m) - \mu t \|\lambda_1 + \cdots + \lambda_m\|^2}}{(\|\lambda_1\| \cdots \|\lambda_m\|)^{(n-\varkappa)/2}} \times W(d\lambda_1) \cdots W(d\lambda_m), \qquad t > 0, \ x \in \mathbb{R}^n, \ 0 < \varkappa < n/m, \ m \geqslant 1, \tag{2.3}$$

where the Tauberian constant

$$c(n, \varkappa) = \Gamma\left(\frac{n-\varkappa}{2}\right) \Big/ [2^{\varkappa} \pi^{n/2} \Gamma(\varkappa/2)], \tag{2.4}$$

and $W(\cdot)$ is the Gaussian complex white noise measure.

Remark 1. The symbol $\int'$ means a multiple stochastic integral with respect to the Gaussian complex white noise measure with the hyperplanes $\lambda_i = \pm \lambda_j$, $i, j = 1,..., m$, $i \neq j$, being excluded from the domain of integration.

Remark 2. For $m \geqslant 1$ and $\varkappa \in (0, n/m)$ the random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, is stationary in x with $EX_m(t, x) = 0$ and covariance function

$$EX_m(t, x)\, X_m(t', y) = R(x-y, t+t').$$

Observe that the field $X_1(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, is Gaussian and stationary in x, with zero mean and spectral density

$$S_{1,2}(\lambda) = C_1^2 c(n, \varkappa) \frac{e^{-\mu(t+t') \|\lambda\|^2}}{\|\lambda\|^{n-\varkappa}}, \qquad \lambda \in \mathbb{R}^n \tag{2.5}$$

such that

$$EX_1(t, x)\, X_1(t', y) = \int_{\mathbb{R}^n} e^{i(\lambda, x-y)} S_{1,2}(\lambda)\, d\lambda.$$

This random field has LRD of the second order, i.e., the spectral density satisfies $S_{1,2}(0) = \infty$.

The random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$ are non-Gaussian with $EX_m^2(t, x) < \infty$. The following theorem describes the second-order spectral density of the non-Gaussian random fields (2.3).

Theorem 2. The random field $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with fixed $t > 0$ and $m \geqslant 2$ is stationary in x, that is,

$$EX_m(t, x)\, X_m(t, x') = \int_{\mathbb{R}^n} e^{i(\lambda, x - x')} S_{m,2}(\lambda)\, d\lambda,$$

with spectral density

$$S_{m,2}(\lambda) = \frac{C_m{}^2}{m!} (c_2(n, \varkappa))^m\, \mathscr{K}(\varkappa, m)\, \frac{e^{-2\mu t\, \|\lambda\|^2}}{\|\lambda\|^{n - m\varkappa}}, \qquad \lambda \in \mathbb{R}^n, \quad 0 < \varkappa < n/m, \tag{2.6}$$

where

$$\mathscr{K}(\varkappa, m) = \pi^{\frac{n}{2}(m-1)} \left\{ \frac{\Gamma(\frac{\varkappa}{2})}{\Gamma(\frac{n-\varkappa}{2})} \right\}^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(\frac{i\varkappa}{2})\, \Gamma(\frac{n-(1+i)\varkappa}{2})}{\Gamma(\frac{n-i\varkappa}{2})\, \Gamma(\frac{(1+i)\varkappa}{2})}. \tag{2.7}$$

The proof of Theorem 2 will be presented in Section 4.

Remark 3. The non-Gaussian random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$ display LRD in space (see (2.6)), that is, $S_{m,2}(0) = \infty$. The rate of convergence to infinity of the spectral density $S_{m,2}(\lambda)$ as $|\lambda| \to 0$ becomes faster and faster when $m \geqslant 2$ increases.

We next describe the bispectra and higher-order spectral densities of the non-Gaussian random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$. Note that the bispectra and all higher-order spectral densities of the Gaussian random field $X_1(t, x)$ are equal to zero. Let us first recall the definition of higher-order spectral densities of real-valued strictly stationary (up to the order $k \geqslant 2$) continuous-parameter random field $Z(x)$, $x \in \mathbb{R}^n$, with $E\,|Z(x)|^k < \infty$, $k \geqslant 2$. Let

$$c_k(x_1,..., x_k) = \frac{1}{i^k} \frac{\partial^k}{\partial u_1 \cdots \partial u_k} \log E \exp\left\{ i \sum_{j=1}^{k} u_j Z(x_j) \right\} \Bigg|_{u_1 = \cdots = u_k = 0}$$

be the cumulant function of a random vector $(Z(x_1),..., Z(x_k))$, $k \geqslant 2$. Then, for strictly stationary fields,

$$c_k(x_1,..., x_k) = c_k(x_1 - x_k,..., x_{k-1} - x_k, 0),$$

and if there exists a complex-valued integrable function

$$S_k(\lambda_1,..., \lambda_{k-1}) \in L_1(\mathbb{R}^{(k-1)\, n})$$

such that

$$c_k(x_1 - x_k, \ldots, x_{k-1} - x_k, 0)$$
$$= \int_{\mathbb{R}^{(k-1)n}} \exp\left\{ i \sum_{j=1}^{k-1} \langle \lambda_j, x_j - x_k \rangle \right\} S_k(\lambda_1, \ldots, \lambda_{k-1})\, d\lambda_1 \cdots d\lambda_{k-1},$$

then $S_k(\lambda_1, \ldots, \lambda_{k-1})$ is called the *spectral density of order* $k \geqslant 2$ of the field $Z(x)$, $x \in \mathbb{R}^n$, (see Ivanov and Leonenko[22] for details).

We should note that $S_k(\lambda_1, \ldots, \lambda_{k-1}) = S_k(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)$ with $\lambda_k = -(\lambda_1 + \cdots + \lambda_{k-1})$. For such a function we will use the following symmetrized version:

$$\operatorname*{sym}_{\{\lambda_1, \ldots, \lambda_k : \lambda_1 + \cdots + \lambda_{k-1} + \lambda_k = 0\}} S_k(\lambda_1, \ldots, \lambda_{k-1}) = \frac{1}{k!} \sum_{\pi \in \mathscr{P}_k} S_k(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(k-1)})$$

where $\mathscr{P}_k$ is the set of all $k!$ permutations $p = (\pi(1), \ldots, \pi(k))$ of the set $\{1, \ldots, k\}$. All our spectral densities of higher-order will be absolutely summable. This will be discussed later in Remark 7.

We are now in a position to formulate our second result which describes the bispectra of the non-Gaussian random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$.

Theorem 3. The random field $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with fixed $t > 0$ and $m \geqslant 2$ is strictly stationary in x of the third order with $E\, |X_m(t, x)|^3 < \infty$. Its bispectra $S_{m,3}(\lambda_1, \lambda_2)$ exist and can be expressed as

$$S_{m,3}(\lambda_1, \lambda_2) = \left(\frac{C_{2k}}{k!} \right)^3 (c(n, \varkappa))^{3k} (\mathscr{K}(n, \varkappa))^3$$
$$\times \operatorname*{sym}_{\{\lambda_1, \lambda_2, \lambda_3 : \lambda_1 + \lambda_2 + \lambda_3 = 0\}} [\exp\{-\mu t (\|\lambda_1\|^2 + \|\lambda_2\|^2 + \|\lambda_1 + \lambda_2\|^2)\}$$
$$\times g_{m,3}(\lambda_1, \lambda_2)], \tag{2.8}$$

$$g_{m,3}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^n} \frac{dz}{(\|\lambda_1 + \lambda_2 + z\|\, \|\lambda_2 + z\|\, \|z\|)^{n - k\varkappa}}, \quad 0 < \varkappa < n/m, \ \lambda_1, \lambda_2 \in \mathbb{R}^n, \tag{2.9}$$

when $m = 2k$, $k = 1, 2, \ldots$ and

$$S_{m,3}(\lambda_1, \lambda_2) = 0,$$

when $m = 2k+1$, $k = 1, 2, \ldots$. The function $g_{m,3}(\lambda_1, \lambda_2)$, $m = 2k$, is homogeneous of order $H = \frac{3}{2} m\varkappa - 2n$, that is, $g_{m,3}(t\lambda_1, t\lambda_2) = t^H g_{m,3}(\lambda_1, \lambda_2)$, and its Fourier transform is given by

$$\hat{g}_{m,3}(\zeta_1, \zeta_2) = \left(\pi^{\frac{n}{2} - k\varkappa} \frac{\Gamma(k\varkappa/2)}{\Gamma((n-k\varkappa)/2)} \right)^3 (\|\zeta_1\| \, \|\zeta_2\| \, \|\zeta_1 - \zeta_2\|)^{-k\varkappa}. \quad (2.10)$$

The proof of Theorem 3 will be given in Section 4.

Remark 4. By Riesz's composition formula (see Appendix A) we obtain that for $\varkappa \in (0, n/m)$

$$g_{m,3}(\lambda_1, 0) = k\left(m\varkappa - n, \frac{m\varkappa}{2} \right) \|\lambda_1\|^{\frac{3}{2} m\varkappa - 2n},$$

$$g_{m,3}(0, \lambda_2) = k\left(m\varkappa - n, \frac{m\varkappa}{2} \right) \|\lambda_2\|^{\frac{3}{2} m\varkappa - 2n},$$

$$g_{m,3}(\lambda, -\lambda) = k\left(m\varkappa - n, \frac{m\varkappa}{2} \right) \|\lambda\|^{\frac{3}{2} m\varkappa - 2n},$$

where

$$k(\alpha, \beta) = \pi^{n/2} \frac{\Gamma(\frac{\alpha}{2}) \, \Gamma(\frac{\beta}{2}) \, \Gamma(\frac{n-\alpha-\beta}{2})}{\Gamma(\frac{n-\alpha}{2}) \, \Gamma(\frac{n-\beta}{2}) \, \Gamma(\frac{\alpha+\beta}{2})}$$

(see Appendix A for the nature of this constant). These formulae express the singular properties of the function $g_{m,3}(\lambda_1, \lambda_2)$, hence of $S_{m,3}(\lambda_1, \lambda_2)$ (see (2.8)), on the hyperplanes $\lambda_2 = 0$, $\lambda_1 = 0$, and $\lambda_1 + \lambda_2 = 0$.

The corresponding trispectra are more complicated. We are able to obtain

Theorem 4. The random field $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with fixed $t > 0$ and $m \geqslant 2$ is strictly stationary in x of the fourth order with $E\,|X_m(t, x)|^4 < \infty$ and its trispectra $S_{m,4}(\lambda_1, \lambda_2, \lambda_3)$ can be expressed as

$$S_{m,4}(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{C_m}{m!} c^{m/2}(n, \varkappa) \right)^4 \sum_{k=1}^{[m/2]} \frac{(m!)^4}{((m-2k)!)^2} 3^k \operatorname*{sym}_{\{\lambda_1, \lambda_2, \lambda_3, \lambda_4 : \sum_{i=1}^4 \lambda_i = 0\}}$$
$$\times \Bigg[\exp\left\{ -\mu t \left(\sum_{i=1}^3 \|\lambda_i\|^2 + \left\| \sum_{i=1}^3 \lambda_i \right\|^2 \right) \right\}$$
$$\times (I_{k,1}(\lambda_1, \lambda_2, \lambda_3) + I_{k,2}(\lambda_1, \lambda_2, \lambda_3) + I_{k,3}(\lambda_1, \lambda_2, \lambda_3)) \Bigg]$$

where

$$I_{k,1}(\lambda_1, \lambda_2, \lambda_3) = \{\mathscr{K}(\varkappa, k)\}^4 \{\mathscr{K}(\varkappa, m-2k)\}^2 \{k(k\varkappa, (m-2k)\varkappa)\}^2$$
$$\times \int_{\mathbb{R}^n} \|\lambda_1+\lambda_2+\mu\|^{k\varkappa-n} \|\lambda_1+\mu\|^{(m-k)\varkappa-n}$$
$$\times \|\lambda_1+\lambda_2+\lambda_3+\mu\|^{(m-k)\varkappa-n} \|\mu\|^{k\varkappa-n}\, d\mu, \tag{2.11}$$

$$I_{k,2}(\lambda_1, \lambda_2, \lambda_3) = \{\mathscr{K}(\varkappa, k)\}^4 \{\mathscr{K}(\varkappa, m-2k)\}^2 \{k(k\varkappa, (m-2k)\varkappa)\}^2$$
$$\times \int_{\mathbb{R}^n} \|\lambda_1+\mu\|^{k\varkappa-n} \|\lambda_1+\lambda_2+\lambda_3+\mu\|^{k\varkappa-n}$$
$$\times \|\lambda_1+\lambda_2+\mu\|^{(m-k)\varkappa-n} \|\mu\|^{(m-k)\varkappa-n}\, d\mu, \tag{2.12}$$

$$I_{k,3}(\lambda_1, \lambda_2, \lambda_3) = \{\mathscr{K}(\varkappa, k)\}^4 \{\mathscr{K}(\varkappa, m-2k)\}^2$$
$$\times \int_{\mathbb{R}^{3n}} \|\lambda_1+\mu-\nu\|^{k\varkappa-n} \|\lambda_1+\lambda_2+\mu-\nu-\lambda\|^{k\varkappa-n}$$
$$\times \|\lambda_1+\lambda_2+\lambda_3+\mu-\lambda\|^{k\varkappa-n}$$
$$\times \|\mu\|^{k\varkappa-n} \|\nu\|^{(m-2k)\varkappa-n} \|\lambda\|^{(m-2k)\varkappa-n}\, d\mu\, d\nu\, d\lambda. \tag{2.13}$$

Remark 5. By Riesz's composition formula (see Appendix A), we obtain that for $\varkappa \in (0, n/m)$

$$I_{k,1}(0, 0, \lambda_3) = K_1 k((m+k)\varkappa-2n, (m-k)\varkappa) \|\lambda_3\|^{2m\varkappa-2n},$$
$$I_{k,1}(0, \lambda_2, 0) = K_1 k(m\varkappa-n, m\varkappa-n) \|\lambda_2\|^{2m\varkappa-2n},$$
$$I_{k,1}(\lambda_1, 0, 0) = K_1 k((2m-k)\varkappa-2n, k\varkappa) \|\lambda_1\|^{2m\varkappa-2n},$$
$$I_{k,1}(\lambda, -\lambda, \lambda) = K_1 k(2k\varkappa-n, 2(m-k)\varkappa-n) \|\lambda\|^{2m\varkappa-2n},$$
$$K_1 = \{\mathscr{K}(\varkappa, k)\}^4 \{\mathscr{K}(\varkappa, m-2k)\}^2 \{k(k\varkappa, (m-2k)\varkappa)\}^2$$

and analogously for $I_{k,2}$:

$$I_{k,2}(0, 0, \lambda_3) = K_1 k((2m-k)\varkappa-2n, k\varkappa) \|\lambda_3\|^{2m\varkappa-2n},$$
$$I_{k,2}(0, \lambda_2, 0) = K_1 k(m\varkappa-n, m\varkappa-n) \|\lambda_2\|^{2m\varkappa-2n},$$
$$I_{k,2}(\lambda_1, 0, 0) = K_1 k((m+k)\varkappa-2n, (m-k)\varkappa) \|\lambda_1\|^{2m\varkappa-2n},$$
$$I_{k,2}(\lambda, -\lambda, \lambda) = K_1 k(2k\varkappa-n, 2(m-k)\varkappa-n) \|\lambda\|^{2m\varkappa-2n}.$$

These formulae describe the singular properties of the functions $I_{k,1}(\lambda_1, \lambda_2, \lambda_3)$ and $I_{k,2}(\lambda_1, \lambda_2, \lambda_3)$ and so of $S_{m,4}(\lambda_1, \lambda_2, \lambda_3)$ on the hyperplanes

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}, \quad \begin{cases} \lambda_1 = 0 \\ \lambda_3 = 0 \end{cases}, \quad \begin{cases} \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}, \quad \begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases}.$$

The spectral densities $S_{m,p}(\lambda_1, \ldots, \lambda_{p-1})$ of order $p > 4$ for the fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, are of a more complicated form. We first consider the cases $m = 2$ and $m = 3$. The spectral densities of an arbitrary order p for these cases are presented in the next two theorems.

Theorem 5. The random field $X_2(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, defined by the representation (2.3) with $m = 2$ and fixed $t > 0$, is strictly stationary in x of order p with $E\,|X_2(t, x)|^p < \infty$. Its spectral densities of order p, $S_{2,p}(\lambda_1, \ldots, \lambda_{p-1})$, can be represented in the form

$$S_{2,p}(\lambda_1, \ldots, \lambda_{p-1}) = \left[\frac{C_2}{2}\, c(n, \varkappa)\right]^p 2^{p-1}(p-1)!$$
$$\times \operatorname*{sym}_{\{\lambda_1, \ldots, \lambda_p : \sum_{i=1}^p \lambda_i = 0\}} \left[e^{-\mu t(\sum_{i=1}^{p-1} \|\lambda_i\|^2 + \|\sum_{i=1}^{p-1} \lambda_i\|^2)} g_{2,p}(\lambda_1, \ldots, \lambda_{p-1})\right], \tag{2.14}$$

where

$$g_{2,p}(\lambda_1, \ldots, \lambda_{p-1}) = \int_{\mathbb{R}^n} \frac{d\lambda}{(\|\lambda\|\, \|\lambda + \lambda_1\| \cdots \|\lambda + \sum_{i=1}^{p-1} \lambda_i\|)^{n-\varkappa}}. \tag{2.15}$$

Remark 6. By Riesz's composition formula (see Appendix A), we obtain the second order spectral density in closed form:

$$S_{2,2}(\lambda) = 2k(\varkappa, \varkappa)\, \|\lambda\|^{2\varkappa - n} \left[\frac{C_2}{2}\, c(n, \varkappa)\right]^2$$

with singularity at $\lambda = 0$.

For $p \geqslant 3$ we obtain

(1) $g_{2,3}(\lambda_1, \lambda_2)$ is of the form $const \times \|\lambda_i\|^{3\varkappa - 2n}$ on the hyperplanes $\{\lambda_1 = 0\}$, $\{\lambda_2 = 0\}$, $\{\lambda_1 = -\lambda_2\}$;

(2) $g_{2,4}(\lambda_1, \lambda_2, \lambda_3)$ is of the form $const \times \|\lambda_i\|^{4\varkappa - 3n}$ on the hyperplanes $\{\lambda_k = \lambda_j = 0\}$, $k \neq j$, $k, j = 1, 2, 3$ and also on the hyperplanes $\{\lambda_k = \lambda_j = -\lambda_i\}$;

(3) $g_{2,5}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is of the form $const \times \|\lambda_l\|^{5\varkappa - 4n}$ on the hyperplanes $\{\lambda_i = \lambda_j = \lambda_k = 0\}$, $i \neq j \neq k \in \{1, 2, 3, 4\}$; $\{\lambda_i = \lambda_j = 0, \ \lambda_k = -\lambda_l\}$, $\{\lambda_i = 0, \lambda_j = \lambda_k = -\lambda_l\}$, $\{\lambda_i = -\lambda_j = \lambda_k = -\lambda_l\}$, $i \neq j \neq k \neq l \in \{1, 2, 3, 4\}$.

And analogously for $p \geqslant 6$.

Theorem 6. The random field $X_3(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, defined by the representation (2.3) with $m = 3$ and fixed $t > 0$ is strictly stationary in x of order p with $E\,|X_3(t, x)|^p < \infty$. Its spectral densities of order p, $S_{3,p}(\lambda_1, \dots, \lambda_{p-1})$, can be represented in the form

$$S_{3,p}(\lambda_1, \dots, \lambda_{p-1}) = \left(\frac{C_3}{3!} (c(n, \varkappa))^{3/2} \right)^p (S_1(\lambda_1, \dots, \lambda_{p-1}) + S_2(\lambda_1, \dots, \lambda_{p-1})) \tag{2.16}$$

for $p = 2k$, $k = 1, 2, \dots$ and

$$S_{3,p}(\lambda_1, \dots, \lambda_{p-1}) = 0$$

for $p = 2k+1$, $k = 1, 2, \dots$, where

$$\begin{aligned} &S_1(\lambda_1, \dots, \lambda_{p-1}) \\ &\quad = (2k-1)! \operatorname*{sym}_{\{\lambda_1, \dots, \lambda_p : \sum_{i=1}^p \lambda_i = 0\}} \Bigg[e^{-\mu t (\sum_{i=1}^{2k-1} \|\lambda_i\|^2 + \|\sum_{i=1}^{2k-1} \lambda_i\|^2)} \\ &\quad \times \sum_{\gamma_1 \in \Gamma^c(1, \dots, 1)} \int_{\mathbb{R}^{n(k+1)}} \prod_{i=1}^{2k-1} \left\| \sum_{j=1}^{i} \lambda_j + \lambda - \sum_{j=1}^{i} \mu_j \right\|^{\varkappa - n} \|\lambda\|^{\varkappa - n} \prod_{j=1}^{2k} \|\mu_j\|^{\varkappa - n} \\ &\quad \times \prod_{(k_i, k_j) \in \mathscr{K}(\gamma_1)} \delta(\mu_{k_i} + \mu_{k_j}) \, d\lambda \, d\lambda\mu_1 \cdots d\lambda\mu_k \Bigg]; \\ &S_2(\lambda_1, \dots, \lambda_{p-1}) \\ &\quad = (2k-1)! \operatorname*{sym}_{\{\lambda_1, \dots, \lambda_p : \sum_{i=1}^p \lambda_i = 0\}} \Bigg[e^{-\mu t (\sum_{i=1}^{2k-1} \|\lambda_i\|^2 + \|\sum_{i=1}^{2k-1} \lambda_i\|^2)} \\ &\quad \times \sum_{\gamma_2 \in \Gamma^c(2, 1, \dots, 1, 2)} \int_{\mathbb{R}^{n(k+1)}} \prod_{i=1}^{2k-1} \left\| \sum_{j=1}^{i} \lambda_j - \mu_{2k+1} - \sum_{j=1}^{i} \mu_j \right\|^{\varkappa - n} \prod_{j=1}^{2k+2} \|\mu_j\|^{\varkappa - n} \\ &\quad \times \prod_{(k_i, k_j) \in \mathscr{K}(\gamma_2)} \delta(\mu_{k_i} + \mu_{k_j}) \, d\mu_1 \cdots d\mu_{k+1} \Bigg]. \end{aligned}$$

In the above formulae, the summation is taken over $\Gamma^c(1, \dots, 1)$, the set of all complete diagrams with $2k$ levels with one vertex in each level, and over $\Gamma^c(2, 1,..., 1, 2)$, the set of all complete diagrams with $2k$ levels with vertices $(2, 1,..., 1, 2)$ and no edge between levels containing 2 vertices; $\mathscr{K}(\gamma)$ denotes the set of edges of the diagrams γ (see Appendix B and the proof of the theorem). In the general case, we can formulate the following result.

Theorem 7. The random fields $X_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $t > 0$ fixed, are strictly stationary in x of order p with $E\,|X_m(t, x)|^p < \infty$. Their spectral densities of order p, $S_{m,p}(\lambda_1,..., \lambda_{p-1})$, can be represented in the form

$$S_{m,p}(\lambda_1,..., \lambda_{p-1}) = \left(\frac{C_m}{m!}\,(c(n, \varkappa))^{m/2}\right)^p (S_1(\lambda_1,..., \lambda_{p-1}) + S_2(\lambda_1,..., \lambda_{p-1})), \tag{2.17}$$

where

$$\begin{aligned} S_1(\lambda_1,..., \lambda_{p-1}) = (p-1)! \underset{\{\lambda_1,...,\lambda_p : \sum_{i=1}^p \lambda_i = 0\}}{\mathrm{sym}} \Bigg[& e^{-\mu t(\sum_{i=1}^{p-1} \|\lambda_i\|^2 + \|\sum_{i=1}^p \lambda_i\|^2)} \\ & \times \sum_{\gamma_1 \in \Gamma_p^c(m-2,...,m-2)} \int_{\mathbb{R}^{n(p(m-2)/2+1)}} \prod_{i=1}^{p-1} \\ & \times \left\| \sum_{j=1}^{i} \lambda_j + \lambda_p - \sum_{j=1}^{i} \sum_{k=(j-1)(m-2)+1}^{j(m-2)} \mu_k \right\|^{\varkappa-n} \\ & \times \|\lambda_p\|^{\varkappa-n} \prod_{k=1}^{p(m-2)} \|\mu_k\|^{\varkappa-n} \prod_{(k_i,k_j) \in \mathscr{K}(\gamma_1)} \{\delta(\mu_{k_i} + \mu_{k_j})\, d\mu_{k_i}\}\, d\lambda_p \Bigg], \end{aligned} \tag{2.18}$$

$$\begin{aligned} S_2(\lambda_1,..., \lambda_{p-1}) = (p-1)! \underset{\{\lambda_1,...,\lambda_p : \sum_{i=1}^p \lambda_i = 0\}}{\mathrm{sym}} \Bigg[& e^{-\mu t(\sum_{i=1}^{p-1} \|\lambda_i\|^2 + \|\sum_{i=1}^p \lambda_i\|^2)} \\ & \times \sum_{\gamma_2 \in \Gamma_p^c(m-1,m-1,m-2,...,m-2)} \int_{\mathbb{R}^{n(p(m-2)/2+1)}} \prod_{i=1}^{p-1} \\ & \times \left\| \sum_{j=1}^{i} \lambda_j - \mu_{p(m-2)+1} - \sum_{j=1}^{i} \sum_{k=(j-1)(m-2)+1}^{j(m-2)} \mu_k \right\|^{\varkappa-n} \\ & \times \prod_{k=1}^{p(m-2)+2} \|\mu_k\|^{\varkappa-n} \prod_{(k_i,k_j) \in \mathscr{K}(\gamma_2)} (\delta(\mu_{k_i} + \mu_{k_j})\, d\mu_{k_i}) \Bigg]. \end{aligned} \tag{2.19}$$

In the above formulae, the summation is taken over $\Gamma^c(m-2,..., m-2)$, the set of all complete diagrams with p levels and $(m-2)$ vertices in each level, and over $\Gamma_p^c(m-1, m-1, m-2, \ldots, m-2)$, the set of all complete diagrams with p levels, 2 levels of which have $m-1$ vertices each and no edge between them, and the rest $p-2$ levels have $m-2$ vertices in each level (see Appendix B and the proof of the theorem).

Remark 7. From Young's inequality we obtain that all the cumulant functions of the above stationary processes with finite moments are bounded. Thus the corresponding spectral densities are integrable. Since they are also real and non-negative, these spectral densities are L_1-functions in corresponding Euclidean spaces.

3. THE RANDOM SPACE-TIME FRACTIONAL HEAT EQUATION

In this section, we present the second-order and higher-order spectral densities for the random fields arising as the limits of the rescaled solutions of the fractional kinetic/diffusion equation

$$\frac{\partial^\beta u}{\partial t^\beta} = -\mu(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u, \qquad \mu>0 \tag{3.1}$$

subject to the random initial conditions

$$u(0,x) = v(x), \qquad x \in \mathbb{R}^n, \tag{3.2}$$

where $\alpha>0$, $\beta \in (0,1]$, $\gamma \geqslant 0$ are fractional parameters, $v(x)$ is a random field of the form $v(x) = h(\xi(x))$, $x \in \mathbb{R}^n$, the non-random function $h(\cdot)$ and the random field $\xi(x)$, $x \in \mathbb{R}^n$, satisfying the conditions A–C introduced in the previous section. Here, Δ is the n-dimensional Laplace operator, and the operators $-(I-\Delta)^{\gamma/2}$, $\gamma \geqslant 0$, and $(-\Delta)^{\alpha/2}$, $\alpha>0$, are interpreted as inverses of the Bessel and Riesz potentials respectively (see Anh and Leonenko[5]). Both Bessel and Riesz potentials are considered to be defined in a weak sense in the frequency domain in terms of fractional Sobolev spaces.

The time derivative of order $\beta \in (0,1]$ is defined as follows:

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \dfrac{\partial u}{\partial t}(t,x), & \text{if } \beta = 1, \\ (\mathscr{D}_t^\beta u)(t,x), & \text{if } \beta \in (0,1), \end{cases}$$

where

$$(\mathscr{D}_t^\beta u)(t,x) = \frac{1}{\Gamma(1-\beta)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\beta} u(\tau,x)\,d\tau - \frac{u(0,x)}{t^\beta}\right], \qquad 0<t\leqslant T,$$

is the regularized fractional derivative or fractional derivative in the Caputo–Djrbashian sense (see Anh and Leonenko[5] for some historical reference).

We will use the following entire function of order $1/\beta$ and type 1:

$$E_\beta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j+1)}, \qquad z \in \mathbb{C}^1, \quad \beta > 0.$$

This function is known as the Mittag–Leffler function (see Djrbashian[15]). In particular, for real $x \geqslant 0$, $\beta > 0$,

$$E_\beta(-x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{\Gamma(\beta j+1)} \tag{3.3}$$

is infinitely differentiable and completely monotonic if $0 < \beta < 1$, that is,

$$(-1)^k \frac{d^k}{dx^k} E_\beta(-x) \geqslant 0, \qquad x \geqslant 0, \quad 0 < \beta < 1, \quad k = 0, 1, 2, \ldots$$

For real $x \geqslant 0$ and $\beta < 1$,

$$E_\beta(-x) = \frac{\sin(\pi\beta)}{\pi\beta} \int_0^\infty \frac{\exp\{-(xt)^{1/\beta}\}\, dt}{t^2 + 2t\cos(\pi\beta) + 1}.$$

In particular, for $x \geqslant 0$, $E_1(-x) = e^{-x}$. For the function E_β, the following asymptotic expansion holds:

$$E_\beta(-x) = -\sum_{k=1}^{N} \frac{(-1)^k x^{-k}}{\Gamma(1-\beta k)} + O(|x|^{-N-1}) \tag{3.4}$$

as $x \to \infty$, where $\beta < 1$ (see Djrbashian,[15] p. 5).

The limiting distributions of the rescaled solutions of the initial value problem (3.1) and (3.2) have been investigated in Anh and Leonenko.[5] Their main result is formulated in the following theorem.

Theorem 8. Let the random field $u(t, x)$, $0 < t \leqslant T$, $x \in \mathbb{R}^n$, be a solution of the fractional diffusion equation (3.1) with the random initial condition (3.2) and assume that the conditions A, B, C are satisfied. Suppose that the Green function of the equation is in $L_1(\mathbb{R}^n)$ and

$$\varkappa < \min(2\alpha, n)/m, \tag{3.5}$$

where the parameter $\varkappa$ is defined in the condition A and $m \geqslant 1$ is the Hermitian rank of the function h. Then the finite-dimensional distributions of the random field

$$U_\varepsilon(t, x) = \frac{1}{\varepsilon^{m\varkappa\beta/(2\alpha)}} u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right), \qquad 0 < t \leqslant T, \quad x \in \mathbb{R}^n,$$

converge weakly as $\varepsilon \to 0$ to the finite-dimensional distributions of the random field

$$U_m(t, x) = \frac{C_m}{m!} c^{m/2}(n, \varkappa) \int'_{\mathbb{R}^{nm}} \frac{e^{i(x, \lambda_1 + \cdots + \lambda_m)}}{(\|\lambda_1\| \cdots \|\lambda_m\|)^{(n-\varkappa)/2}} E_\beta(-\mu t^\beta \|\lambda_1 + \cdots + \lambda_m\|^\alpha)$$
$$\times W(d\lambda_1) \cdots W(d\lambda_m), \qquad 0 < t \leqslant T, \quad x \in \mathbb{R}^n, \tag{3.6}$$

where E_β is the Mittag–Leffler function (3.3), W is the complex Gaussian white noise measure and $c(n, \varkappa)$ is a constant defined by (2.4). Here, $\int'$ is the multiple Wiener–Itô integral with respect to a complex Gaussian white noise measure $W(\cdot)$ with the diagonal hyperplanes $\lambda_i = \pm\lambda_j$, $i, j = 1,..., m$, $i \neq j$, being excluded from the domain of integration.

Remark 8. For $m \geqslant 1$ and $\varkappa < \min(2\alpha, n)/m$, the random field $U_m(t, x)$ is stationary in $x \in \mathbb{R}^n$ with covariance function

$$EU_m(t, x)\, U_m(s, y) = \frac{C_m^2}{m!} c^m(n, \varkappa) \int'_{\mathbb{R}^{nm}} \frac{e^{i(x-y, \lambda_1 + \cdots + \lambda_m)}}{(\|\lambda_1\| \cdots \|\lambda_m\|)^{n-\varkappa}}$$
$$\times E_\beta(-\mu t^\beta \|\lambda_1 + \cdots + \lambda_m\|^\alpha)$$
$$\times E_\beta(-\mu s^\beta \|\lambda_1 + \cdots + \lambda_m\|^\alpha)\, d\lambda_1 \cdots d\lambda_m. \tag{3.7}$$

For $\varkappa < \min(2\alpha, n)/m$, we have $EU_m^2(t, x) < \infty$.

Observe that $U_1(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, is a stationary (in x) Gaussian random field with covariance function (3.7) with $m = 1$ and spectral density

$$S_{1,2}(\lambda) = C_1^2 c(n, \varkappa)\, E_\beta^2(-\mu t^\beta \|\lambda\|^\alpha)\, \|\lambda\|^{\varkappa - n}, \qquad \lambda \in \mathbb{R}^n. \tag{3.8}$$

The spectral density (3.8) behaves as

$$C_1^2 c(n, \varkappa)\, \|\lambda\|^{\varkappa - n}, \qquad \varkappa \in (0, \min(2\alpha, n))$$

as $\|\lambda\| \to 0$. Hence the Gaussian random field $U_1(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, which can be considered as an approximation to the solution of the fractional kinetic equation with random singular data, displays LRD.

Note also that as $\|\lambda\| \to \infty$

$$S_{1,2}(\lambda) = \frac{C_1^2 c(n, \varkappa)}{\mu t^\beta} \frac{1}{\|\lambda\|^{n+2\alpha-\varkappa}} + O\left(\frac{1}{\|\lambda\|^{n+2\alpha-\varkappa+1}}\right)$$

(see Anh and Leonenko[5]). The component $1/\|\lambda\|^{n+2\alpha-\alpha}$ indicates the second-order intermittency of the random field (see Anh *et al.*[1]).

Remark 9. Theorem 8 reduces to Theorem 1 for $n \geqslant 1$, $\beta = 1$, $\gamma = 0$, $\alpha = 2$.

The second-order and higher-order spectral densities for the non-Gaussian random fields $U_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$ are given in the next two theorems.

Theorem 9. Consider the random field $U_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with fixed $t > 0$ and $m \geqslant 2$. This random field is stationary in x, that is,

$$EU_m(t, x)\, U_m(t, x') = \int_{\mathbb{R}^n} e^{i(\lambda, x-x')} S_{U_m,2}(\lambda)\, d\lambda$$

with the following spectral density

$$S_{U_m,2}(\lambda) = \frac{C_m^2}{m!}\, c^m(n, \varkappa)\, \mathscr{K}(\varkappa, m)(E_\beta(-\mu t^\beta \|\lambda\|^\alpha))^2 \|\lambda\|^{m\varkappa-n}. \tag{3.9}$$

Remark 10. The non-Gaussian random fields $U_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $m \geqslant 2$ display LRD in space. In fact, from (3.9) we have that their spectral density behaves as

$$const \times \|\lambda\|^{m\varkappa-n}, \qquad \varkappa \in (0, \min(2\alpha, n))$$

as $\|\lambda\| \to 0$.

From (3.4) we also conclude that

$$S_{U_m,2}(\lambda) = \left(\frac{C_m}{m!}\right)^2 \frac{c^m(n, \varkappa) \times 2\mathscr{K}(\varkappa, m)}{\mu t^\beta} \frac{1}{\|\lambda\|^{n+2\alpha-m\varkappa}} + O\left(\frac{1}{\|\lambda\|^{n+2\alpha-m\varkappa+1}}\right).$$

In the next theorem we describe higher-order spectra for the fields $U_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$.

Theorem 10. The random fields $U_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, with $t > 0$ fixed are strictly stationary in x of order p with $E\,|U_m(t, x)|^p < \infty$, and their spectral densities can be represented in the following forms:

(1) The bispectra

$$S_{U_m, 3}(\lambda_1, \lambda_2) = \left(\frac{C_{2k}}{k!}\right)^3 (c(n, \varkappa))^{3k} (\mathscr{K}(\varkappa, m))^3 \underset{\{\lambda_1, \lambda_2, \lambda_3 : \sum_{i=1}^{3} \lambda_i = 0\}}{\mathrm{sym}} \times \Big[E_\beta(-\mu t^\beta \|\lambda_1\|^\alpha)\, E_\beta(-\mu t^\beta \|\lambda_2\|^\alpha)\, E_\beta(-\mu t^\beta \|\lambda_1 + \lambda_2\|^\alpha) \times \int_{\mathbb{R}^n} (\|\lambda_1 + \lambda_2 + \lambda\|\, \|\lambda_2 + \lambda\|\, \|\lambda\|)^{k\varkappa - n}\, d\lambda \Big] \quad \text{for} \quad m = 2k$$

and

$$S_{U_m, 3}(\lambda_1, \lambda_2) = 0 \qquad \text{for} \quad m = 2k+1.$$

(2) The trispectra

$$S_{U_m, 4}(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{C_m}{m!}\, c^{m/2}(n, \varkappa)\right)^4 \sum_{k=1}^{[m/2]} \frac{(m!)^4\, 3^{k+1}}{((m-2k)!)^2} \times \underset{\{\lambda_1, \lambda_2, \lambda_3, \lambda_4 : \sum_{i=1}^{4} \lambda_i = 0\}}{\mathrm{sym}} \Big[\prod_{i=1}^{3} E_\beta(-\mu t^\beta \|\lambda_i\|^\alpha) \times E_\beta\Big(-\mu t^\beta \Big\|\sum_{i=1}^{3} \lambda_i\Big\|^\alpha\Big) \{I_{k,1} + I_{k,2} + I_{k,3}\} \Big],$$

where $I_{k,1}$, $I_{k,2}$, $I_{k,3}$ are given by formulae (2.11)–(2.13).

(3) The spectral densities of order p can be written in the form

$$S_{U_m, p}(\lambda_1, \ldots, \lambda_{p-1}) = \left(\frac{C_m}{m!}\, (c(n, \varkappa))^{m/2}\right)^p (S^{(1)}_{U_m}(\lambda_1, \ldots, \lambda_{p-1}) + S^{(2)}_{U_m}(\lambda_1, \ldots, \lambda_{p-1})),$$

where

$$S^{(1)}_{U_m}(\lambda_1, \ldots, \lambda_{p-1}) = (p-1)! \underset{\{\lambda_1, \ldots, \lambda_p : \sum_{i=1}^{p} \lambda_i = 0\}}{\mathrm{sym}} \Big[\prod_{i=1}^{p-1} E_\beta(-\mu t^\beta \|\lambda_i\|^\alpha) \times E_\beta\Big(-\mu t^\beta \Big\|\sum_{i=1}^{p-1} \lambda_i\Big\|^\alpha\Big) \Sigma_1(\lambda_1, \ldots, \lambda_{p-1}) \Big]$$

and

$$S^{(2)}_{U_m}(\lambda_1,\dots,\lambda_{p-1}) = (p-1)! \underset{\{\lambda_1,\dots,\lambda_p:\sum_{i=1}^{p}\lambda_i=0\}}{\mathrm{sym}} \left[\prod_{i=1}^{p-1} E_\beta(-\mu t^\beta \|\lambda_i\|^\alpha) \times E_\beta\left(-\mu t^\beta \left\|\sum_{i=1}^{p-1}\lambda_i\right\|^\alpha\right)\Sigma_2(\lambda_1,\dots,\lambda_{p-1})\right].$$

Here, $\Sigma_1(\lambda_1,\dots,\lambda_{p-1})$ and $\Sigma_2(\lambda_1,\dots,\lambda_{p-1})$ are equal to the sums appearing on the right-hand side of the formulae (2.18) and (2.19) respectively.

Remark 11. The structure of the set of singularities of the higher-order spectral densities described in Theorems 9 and 10 are similar to that of the set of singularities of the spectral densities from Section 2 because they are defined via the same integrals (see Remarks 2–6). However, the behaviour of the spectral densities in Theorems 9 and 10 at infinity is quite different from that of the spectral densities of Section 2: Here, they have power-type behaviour which indicates higher-order intermittency.

4. PROOFS

We provide in this section the proofs for Theorems 2 and 3 to illustrate the techniques, particularly the use of the Riesz composition formula and the diagram formula for evaluating the products of multiple stochastic Wiener–Itô integrals. These formulae are given in Appendices A and B below. The proofs of the remaining theorems, although more involved, can be constructed in a similar fashion. Complete details of these proofs are provided in Anh *et al.*[7]

Proof of Theorem 2. Consider

$$\mathrm{Cov}(X_m(t,x), X_m(t,x+y))$$

$$= \frac{C_m^2}{m!}(c(n,\varkappa))^m \int_{\mathbb{R}^{nm}} \frac{e^{i(y,\lambda_1+\cdots+\lambda_m)-2\mu t\|\lambda_1+\cdots+\lambda_m\|^2}}{(\|\lambda_1\|\cdots\|\lambda_m\|)^{n-\varkappa}}\, d\lambda_1\cdots d\lambda_m$$

The change of variables $\lambda_1=\lambda_1'-\lambda_2'$, $\lambda_2=\lambda_2'-\lambda_3',\dots,$ $\lambda_{m-1}=\lambda_{m-1}'-\lambda_m'$, $\lambda_m=\lambda_m'$ yields $\sum_{i=1}^m \lambda_i=\lambda_1'$ and

$$\int_{\mathbb{R}^{nm}} e^{i(y,\,\lambda_1+\cdots+\lambda_m)} \frac{e^{-2\mu t\,\|\lambda_1+\cdots+\lambda_m\|^2}}{(\|\lambda_1\|\cdots\|\lambda_m\|)^{n-\varkappa}}\,d\lambda_1\cdots d\lambda_m$$

$$=\int_{\mathbb{R}^{nm}} e^{i(y,\,\lambda_1)} \frac{e^{-2\mu t\,\|\lambda_1\|^2}}{(\|\lambda_1-\lambda_2\|\cdots\|\lambda_{m-1}-\lambda_m\|\,\|\lambda_m\|)^{n-\varkappa}}\,d\lambda_1\cdots d\lambda_m$$

$$=\int_{\mathbb{R}^{nm}} e^{i(y,\,\lambda)}\left(e^{-2\mu t\,\|\lambda\|^2}\int_{\mathbb{R}^{(m-1)\,n}}\frac{d\lambda_2\cdots d\lambda_m}{(\|\lambda-\lambda_2\|\,\|\lambda_2-\lambda_3\|\cdots\|\lambda_{m-1}-\lambda_m\|\,\|\lambda_m\|)^{n-\varkappa}}\right)d\lambda.$$

It follows that the spectral density of $X_m(t, x)$ is given by

$$S_{m,\,2}(\lambda)=e^{-2\mu t\,\|\lambda\|^2}\int_{\mathbb{R}^{(m-1)\,n}}\frac{d\lambda_2\cdots d\lambda_m}{(\|\lambda-\lambda_2\|\,\|\lambda_2-\lambda_3\|\cdots\|\lambda_{m-1}-\lambda_m\|\,\|\lambda_m\|)^{n-\varkappa}}$$

$$\times\frac{C_m^2}{m!}\,(c(n,\varkappa))^m.$$

Let us denote $f(x)=1/\|x\|^{n-\varkappa}$, $f^{*m}(x)=\int_{\mathbb{R}^{(m-1)\,n}} f(x-\lambda_2)\,f(\lambda_2-\lambda_3)\cdots f(\lambda_{m-1}-\lambda_m)\,f(\lambda_m)\,d\lambda_2\cdots d\lambda_m$; then we can rewrite the expression for $S_{m,\,2}$ in the form

$$S_{m,\,2}(\lambda)=e^{-2\mu t\,\|\lambda\|}f^{*m}(\lambda)\,\frac{C_m^2}{m!^2}\,(c(n,\varkappa))^m.$$

The convolution $f^{*m}(\lambda)$ can be written out in a closed form if we apply the Riesz composition formula (see Appendix A). In fact,

$$f^{*m}(\lambda)=\int_{\mathbb{R}^{(m-1)\,n}} f(\lambda-\lambda_2)\,f(\lambda_2-\lambda_3)\cdots f(\lambda_{m-1}-\lambda_m)\,f(\lambda_m)\,d\lambda_2\cdots d\lambda_m$$

$$=\int_{\mathbb{R}^{(m-1)\,n}}\frac{d\lambda_2\cdots d\lambda_m}{(\|\lambda-\lambda_2\|\,\|\lambda_2-\lambda_3\|\cdots\|\lambda_{m-1}-\lambda_m\|\,\|\lambda_m\|)^{n-\varkappa}},\qquad 0<m\varkappa<n.$$

From the Riesz formula,

$$\int_{\mathbb{R}^n}\frac{d\lambda_m}{(\|\lambda_{m-1}-\lambda_m\|\,\|\lambda_m\|)^{n-\varkappa}}=k(\varkappa,\varkappa)\,\|\lambda_{m-1}\|^{2\varkappa-n}.$$

Hence we can write

$$f^{*m}(\lambda) = \int_{\mathbb{R}^n} \|\lambda - \lambda_2\|^{\varkappa - n} \int_{\mathbb{R}^n} \|\lambda_2 - \lambda_3\|^{\varkappa - n}$$

$$\times \int_{\mathbb{R}^n} \|\lambda_{m-1} - \lambda_m\|^{\varkappa - n} \|\lambda_m\|^{\varkappa - n} \, d\lambda_m \, d\lambda_{m-1} \cdots d\lambda_2$$

$$= k(\varkappa, \varkappa) \, k(\varkappa, 2\varkappa) \cdots k(\varkappa, (m-2)\,\varkappa) \, k(\varkappa, (m-1)\,\varkappa) \, \|\lambda\|^{m\varkappa - n}$$

$$= \mathcal{K}(\varkappa, m) \, \|\lambda\|^{m\varkappa - n},$$

where

$$\mathcal{K}(\varkappa, m) = \pi^{\frac{n}{2}(m-1)} \left\{ \frac{\Gamma(\frac{\varkappa}{2})}{\Gamma(\frac{n-\varkappa}{2})} \right\}^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(\frac{i\varkappa}{2}) \, \Gamma(\frac{n-(1+i)\varkappa}{2})}{\Gamma(\frac{n-i\varkappa}{2}) \, \Gamma(\frac{(1+i)\varkappa}{2})}.$$

So we have

$$f^{*m}(\lambda) = \mathcal{K}(\varkappa, m) \, \|\lambda\|^{m\varkappa - n}$$

and

$$S_{m,2}(\lambda) = e^{-2\mu\|\lambda\|} \, \|\lambda\|^{m\varkappa - n} \, \mathcal{K}(\varkappa, m) \, \frac{C_m^2}{m!} \, (c(n, \varkappa))^m. \quad \blacksquare$$

Proof of Theorem 3. To calculate the bispectrum, we consider the cumulant

$$\mathrm{Cum}(X_m(t, x_1), X_m(t, x_2), X_m(t, x_3)) = c_3 \sum_{\gamma \in \Gamma^c_{3,m}} h_\gamma, \qquad c_3 = \left(\frac{C_m}{m!} \, (c(n, \varkappa))^{m/2} \right)^3, \tag{4.1}$$

$\Gamma^c_{3,m}$ being the set of complete closed diagrams with 3 levels $\{n_1, n_2, n_3\} = \{m, m, m\}$, with $\Gamma^c_{3,m} = \varnothing$ if $m = 2k+1$. So the cumulant is equal to zero if m is odd.

Let us now consider the case $m = 2k$. Denote

$$h_x(\lambda_1, \ldots, \lambda_m) = \frac{e^{i(x, \lambda_1 + \cdots + \lambda_m) - \mu t \|\lambda_1 + \cdots + \lambda_m\|^2}}{(\|\lambda_1\| \cdots \|\lambda_m\|)^{(n-\varkappa)/2}} \tag{4.2}$$

and for a diagram $\gamma \in \Gamma^c_{3,m}$, $m = 2k$, let $\mathscr{K}(\gamma) = \{(k_i, k_j)\}$ be the set of edges; then h_γ can be written in the form

$$h_\gamma = \int_{\mathbb{R}^{n3k}} h_{x_1}(\lambda_1,..., \lambda_{2k})\, h_{x_2}(\lambda_{2k+1},..., \lambda_{4k})\, h_{x_3}(\lambda_{4k+1},..., \lambda_{6k})$$

$$\times \prod_{(k_i, k_j) \in \mathscr{K}(\gamma)} \delta(\lambda_{k_i} + \lambda_{k_j}) \prod_{i=1}^{3k} d\lambda_{k_i}.$$

The restriction that the diagrams $\gamma \in \Gamma^c_{3,m}$ be complete and closed leads to the following construction of the diagrams: for each level of the diagram, say, the first, one half of its vertices are connected to the second level and the rest of the vertices are connected to the third level. Note also that $|\Gamma^c_{3,m}| = ((2k)!/k!)^3$. It follows that

$$h_\gamma = \int_{\mathbb{R}^{n3k}} h_{x_1}(\lambda_1 + \cdots + \lambda_k - \lambda_{k+1} - \cdots - \lambda_{2k})$$

$$\times h_{x_2}(\lambda_{k+1} + \cdots + \lambda_{2k} - \lambda_{2k+1} - \cdots - \lambda_{3k})$$

$$\times h_{x_3}(-\lambda_1 - \cdots - \lambda_k + \lambda_{2k+1} + \cdots + \lambda_{3k})\, d\lambda_1 \cdots d\lambda_{3k}$$

$$= \int_{\mathbb{R}^{n3k}} e^{i[(x_1, \sum_{i=1}^k \lambda_i - \sum_{i=k+1}^{2k} \lambda_i) + (x_2, \sum_{i=k+1}^{2k} \lambda_i - \sum_{i=2k+1}^{3k} \lambda_i) + (x_3, -\sum_{i=1}^k \lambda_i + \sum_{i=2k+1}^{3k} \lambda_i)]}$$

$$\times e^{-\mu t[\|\sum_{i=1}^k \lambda_i - \sum_{i=k+1}^{2k} \lambda_i\|^2 + \|\sum_{i=k+1}^{2k} \lambda_i - \sum_{i=2k+1}^{3k} \lambda_i\|^2 + \|-\sum_{i=1}^k \lambda_i + \sum_{i=2k+1}^{3k} \lambda_i\|^2]}$$

$$\times \frac{d\lambda_1 \cdots d\lambda_{3k}}{(\|\lambda_1\| \cdots \|\lambda_{3k}\|)^{n-\varkappa}}.$$

With the change of variables:

$$\begin{aligned}
\textstyle\sum_{i=1}^k \lambda_i - \sum_{i=k+1}^{2k} \lambda_i &= \omega_1 \\
\textstyle\sum_{i=k+1}^{2k} \lambda_i - \sum_{i=2k+1}^{3k} \lambda_i &= \omega_2 \\
\textstyle\sum_{i=2k+1}^{3k} \lambda_i &= \lambda'_{3k} \\
\textstyle\sum_{i=1}^{p} \lambda_i &= \lambda'_p, \qquad p = 1, 2,..., k-1 \\
\textstyle\sum_{i=k+1}^{p} \lambda_i &= \lambda'_p, \qquad p = k+1, k+2,..., 2k-1 \\
\textstyle\sum_{i=2k+1}^{p} \lambda_i &= \lambda'_p, \qquad p = 2k+1, 2k+2,..., 3k-1,
\end{aligned} \tag{4.3}$$

that is,

$$\lambda_i = \lambda'_i, \qquad i = 1, k+1, 2k+1$$

$$\lambda_k = \omega_1 + \omega_2 + \lambda'_{3k} - \lambda'_{k-1}$$

$$\lambda_{2k} = \omega_2 + \lambda'_{3k} - \lambda'_{2k-1}$$

and for the remaining variables $\lambda_i = \lambda'_{i+1} - \lambda'_i$, $i \notin \{1, k+1, 2k+1, k, 2k\}$, the integral is transformed into the following

$$\int_{\mathbb{R}^{n\cdot 2}} e^{i(x_1 - x_3, \omega_1) + i(x_2 - x_3, \omega_2)} e^{-\mu t(\|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_1 + \omega_2\|^2)}$$

$$\times \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^{n(k-1)}} f(\lambda_1) \prod_{i=2}^{k-1} f(\lambda_i - \lambda_{i-1})\, f(\omega_1 + \omega_2 + \lambda_{3k} - \lambda_{k-1})\, d\lambda_1 \cdots d\lambda_{k-1} \right.$$

$$\times \int_{\mathbb{R}^{n(k-1)}} f(\lambda_{k+1}) \prod_{i=k+2}^{2k-1} f(\lambda_i - \lambda_{i-1})\, f(\omega_1 + \lambda_{3k} - \lambda_{2k-1})\, d\lambda_{k+1} \cdots d\lambda_{2k-1}$$

$$\left. \times \int_{\mathbb{R}^{n(k-1)}} f(\lambda_{2k+1}) \prod_{i=2k+2}^{3k-1} f(\lambda_i - \lambda_{i-1})\, f(\lambda_{3k} - \lambda_{3k-1})\, d\lambda_{2k+1} \cdots d\lambda_{3k-1} \right]$$

$$\times d\lambda_{3k}\, d\omega_1\, d\omega_2$$

$$= \int_{\mathbb{R}^{n\cdot 2}} e^{i[(x_1 - x_3, \omega_1) + (x_2 - x_3, \omega_2)]} e^{-\mu t(\|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_1 + \omega_2\|^2)}$$

$$\times \int_{\mathbb{R}^n} f^{*k}(\omega_1 + \omega_2 + \lambda_{3k})\, f^{*k}(\omega_2 + \lambda_{3k})\, f^{*k}(\lambda_{3k})\, d\lambda_{3k}\, d\omega_1\, d\omega_2.$$

The number of terms in the sum on the right-hand side of (4.1) is $((2k)!/k!)^3$, hence we arrive at the following expression for the bispectrum of the process $X_m(t, x)$:

$$S_{m,3}(\omega_1, \omega_2) = c_3 \left(\frac{(2k)!}{k!} \right)^3 \underset{\{\omega_1, \omega_2, \omega_3 : \Sigma_{i=1}^3 \omega_i = 0\}}{\text{sym}} [e^{-\mu t(\|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_1 + \omega_2\|^2)}]$$

$$\times \int_{\mathbb{R}^n} f^{*k}(\omega_1 + \omega_2 + \lambda)\, f^{*k}(\omega_2 + \lambda)\, f^{*k}(\lambda)\, d\lambda, \qquad \text{for} \quad m = 2k,$$

where

$$\int_{\mathbb{R}^n} f^{*k}(\omega_1+\omega_2+\lambda)\, f^{*k}(\omega_2+\lambda)\, f^{*k}(\lambda)$$

$$=\{\mathscr{K}(\varkappa, k)\}^3 \int_{\mathbb{R}^n} (\|\omega_1+\omega_2+\lambda\|\, \|\omega_2+\lambda\|\, \|\lambda\|)^{k\varkappa-n}\, d\lambda$$

and the constant $c_3(2k!/k!)^3 = (C_{2k}/k!)^3\,(c(n,\varkappa))^{3k}$. For $m=2k+1$, $S_{m,3}=0$. ▮

APPENDIX A. RIESZ'S COMPOSITION FORMULA

The following statement is known as Riesz's composition formula: Suppose that $0<\alpha<n$, $0<\beta<n$, $0<\alpha+\beta<n$, then

$$\int_{\mathbb{R}^n} \|x-z\|^{\alpha-n}\, \|x-y\|^{\beta-n}\, dz = k(\alpha,\beta)\, \|x-y\|^{\alpha+\beta-n},$$

where

$$k(\alpha,\beta)=\pi^{n/2}\, \frac{\Gamma(\frac{\alpha}{2})\,\Gamma(\frac{\beta}{2})\,\Gamma(\frac{n-\alpha-\beta}{2})}{\Gamma(\frac{n-\alpha}{2})\,\Gamma(\frac{n-\beta}{2})\,\Gamma(\frac{\alpha+\beta}{2})} \tag{A.1}$$

(see du Plessis,[18] p. 71).

APPENDIX B. CUMULANTS OF MULTIPLE STOCHASTIC INTEGRALS

This Appendix is based on Dobrushin,[16] Taqqu,[40] Fox and Taqqu,[19] and Terdik.[42]

One of the basic tools for evaluating products of multiple stochastic Wiener–Itô integrals and their moments is the diagram formula. It originates from the diagram formula for the products of Hermite polynomials of Gaussian random variables. We prepare here the formula for evaluating the cumulants of multiple stochastic integrals which is a consequence of the diagram formula.

We first introduce some notations and definitions.

Let $m_1,\ldots, m_p$ be given positive integers. An undirected graph Γ with $m_1+\cdots+m_p=M$ vertices is called a diagram of order $(m_1,\ldots, m_p)$ if

(a) the set of vertices V of the graph Γ is of the form

$$V = \{(1,1),..., (1,m_1), (2,1),..., (2,m_2),..., (p,1),..., (p,m_p)\} = \bigcup_{j=1}^{p} W_j, \tag{B.1}$$

where

$$W_j = \{(j,l): 1 \leqslant l \leqslant n_j\}$$

is the jth level of the graph Γ, $1 \leqslant j \leqslant p$;

(b) each vertex is at most of degree 1, that is, met by at most one edge;

(c) if vertices (j_1, i_1) and (j_2, i_2) are joined by an edge $w = ((j_1, i_1), (j_2, i_2))$, then $j_1 \neq j_2$, that is, the edges of the graph Γ can connnect only different levels.

Let $\Gamma(m_1,..., m_p)$ denote the set of diagrams of order $(m_1,..., m_p)$. Denote by $\mathscr{K}(\gamma)$ the set of edges of a diagram $\gamma \in \Gamma(m_1,..., m_p)$. With each element $v \in V$, we can associate the integer denoting the position at which v appears at the list (B.1). Thus the position of $(1,1)$ is 1, the positions of $(1,2)$ is 2 and so on. The position of the last vertex (p, m_p) is M. Each edge $w = ((j_1, i_1), (j_2, i_2)) \in \mathscr{K}(\gamma)$ can also be thought of as $w = (k_1, k_2)$, where k_1 is the position of the vertex (j_1, i_1) and k_2 is the position of the vertex (j_2, i_2) in the list (B.1). A diagram γ is called *complete* if each of its vertices is met by an edge, that is, there exists no isolated vertices. In such a case, the number of edges in γ is $|\mathscr{K}(\gamma)| = M/2$. A diagram is called *closed* if the set of its levels $\{W_j, j = 1,..., p\}$ cannot be split into two subsets connected by no edge.

Let $h_i \in L_2(\mathbb{R}^{nm_i})$, $i = 1,..., p$, and define

$$h(\lambda_1,..., \lambda_M) = \prod_{i=1}^{p} h_i(\lambda_{M_{i-1}+1},..., \lambda_{M_i}),$$

where $M_i = m_1 + \cdots + m_i$, $i = 1, 2,..., p$, $M_0 = 0$, and $M_p = M$. The following formula is used extensively in this paper:

$$\operatorname{Cum}\left(\int_{\mathbb{R}^{nm_1}}' h_1(\lambda_1,..., \lambda_{m_1}) \prod_{i=1}^{m_1} W(d\lambda_i),..., \int_{\mathbb{R}^{nm_p}}' h_p(\lambda_1,..., \lambda_{m_p}) \prod_{i=1}^{m_p} W(d\lambda_i)\right)$$

$$= \sum_{\gamma \in \Gamma^c(m_1,..., m_p)} \int_{\mathbb{R}^{nM/2}} h(\lambda_1,..., \lambda_M) \prod_{(k_i, k_j) \in \mathscr{K}(\gamma)} \{\delta(\lambda_{k_i} + \lambda_{k_j})\, d\lambda_{k_i}\}, \tag{B.2}$$

where the sum is over all complete closed diagrams γ of order $(m_1,\ldots, m_p)$, $\mathcal{K}(\gamma)$ is the set of edges of the diagrams γ, and $\delta(\cdot)$ is the Kronecker delta function.

ACKNOWLEDGMENTS

Partially supported by the Australian Research Council Grants A89601825 and A10024117, and the NATO Grant PST.CLG.976361.

REFERENCES

1. V. V. Anh, J. M. Angulo, and M. D. Ruiz-Medina, Possible long-range dependence in fractional random fields, *J. Statist. Plann. Inference* **80**:95–110 (1999).
2. V. V. Anh, C. C. Heyde, and Q. Tieng, Stochastic models for fractal processes, *J. Statist. Plann. Inference* **80**:123–135 (1999).
3. V. V. Anh and N. N. Leonenko, Non-Gaussian scenarios for the heat equation with singular initial data, *Stochastic Process. Appl.* **84**:91–114 (1999).
4. V. V. Anh and N. N. Leonenko, Scaling laws for fractional diffusion-wave equation with singular initial data, *Statist. Probab. Lett.* **48**:239–252 (2000).
5. V. V. Anh and N. N. Leonenko, Spectral analysis of fractional kinetic equations with random data, *J. Statist. Phys.* **104**:1349–1387 (2001).
6. V. V. Anh and N. N. Leonenko, Renormalization and homogenization of fractional diffusion equations with random data, *Probab. Theory Related Fields*, (2002), in press.
7. V. V. Anh, N. N. Leonenko, and L. M. Sakhno, Higher-order spectral densities of fractional random fields, *Mathematical Report 07/02, Centre in Statistical Science and Industrial Mathematics, QUT*, (2002).
8. O. E. Barndorff-Nielsen and K. Prause, Apparent scaling, *Finan. Stoch.* **5**:103–113 (2001).
9. O. E. Barndorff-Nielsen and N. Shephard, Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics, part 2, *J. Roy. Statist. Soc. Ser. B* **63**:167–241 (2001).
10. S. I. Boyarchenko and S. Z. Levendorskii, Non-Gaussian Merton–Black–Scholes theory, in *Advanced Series on Statistical Science and Applied Probability*, Vol. 9 (World Scientific, 2002).
11. D. R. Brillinger, Introduction to polyspectra, *Ann. Math. Statist.* **36**:1351–1374 (1965).
12. D. R. Brillinger, Fourier inference: Some methods for the analysis of array and nongaussian series data, *Water Resources Bull.* **21**:743–756 (1985).
13. D. R. Brillinger and M. Rosenblatt, Asymptotic theory of estimates of kth order spectra, in *Spectral Analysis of Time Series*, B. Harris, ed. (Wiley, New York, 1967), pp. 153–188.
14. D. R. Brillinger and M. Rosenblatt, Computation and interpretation of kth order spectra, in *Spectral Analysis of Time Series*, B. Harris, ed. (Wiley, New York, 1967), pp. 189–232.
15. M. M. Djrbashian, *Harmonic Analysis and Boundary Value Problems in Complex Domain* (Birkhäuser Verlag, Basel, 1993).
16. R. L. Dobrushin, Gaussian and their subordinated self-similar random generalized fields, *Ann. Probab.* **7**:1–28 (1979).
17. R. L. Dobrushin and P. Major, Non-central limit theorem for non-linear functionals of Gaussian fields, *Z. Wahrsch. Verw. Gebiete* **50**:1–28 (1979).
18. N. du Plessis, *An Introduction to Potential Theory* (Oliver & Boyd, Edinburgh, 1970).

19. R. Fox and M. S. Taqqu, Multiple stochastic integrals with dependent integrators, *J. Multivariate Anal.* **21**:105–127 (1987).
20. U. Frisch, *Turbulence* (Cambridge University Press, 1995).
21. R. Hilfer, Fractional time evolution, in *Applications of Fractional Calculus in Physics*, R. Hilfer, ed. (World Scientific, Singapore, 2000), pp. 87–130.
22. A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields* (Kluwer, Dordrecht, 1989).
23. S. Jaffard, The multifractal nature of Lévy processes, *Probab. Theory Related Fields* **114**:207–227 (1999).
24. S. Jaffard, On the Frisch–Parisi conjecture, *J. Math. Pures Appl.* **79**:525–552 (2000).
25. J. Kampé de Feriet, Random solutions of the partial differential equations, in *Proc. 3rd Berkeley Symp. Math. Stat. Prob., Volume III* (University of California Press, Berkeley, CA, 1955), pp. 199–208.
26. A. N. Kochubei, Fractional order diffusion, *J. Differential Equations* **26**:485–492 (1990).
27. N. N. Leonenko and W. Woyczynski, Scaling limits of solution of the heat equation with non-Gaussian data, *J. Statist. Phys.* **91**:423–428 (1998).
28. F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, *Appl. Math. Lett.* **9**:23–28 (1996).
29. F. Mainardi and R. Gorenflo, On Mittag–Leffler type functions in fractional evolution processes, *J. Comput. Appl. Math.* **118**:283–299 (2000).
30. F. Mainardi, Y. Luchko, and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fractional Calculus and Applied Analysis* **4**:153–192 (2001).
31. R. Metzler and J. Krafter, The random walk's guide to anomalous diffusion: a fractional dynamic approach, *Phys. Rep.* **339**:1–72 (2000).
32. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
33. M. B. Priestley, *Non-linear and Non-stationary Time Series Analysis* (Academic Press, London, 1988).
34. R. H. Riedi, *Multifractal Processes*, Technical Report TR99-06 (ECE Dept., Rice University, 1999).
35. M. Rosenblatt, Remark on the Burgers equation, *J. Math. Phys.* **9**:1129–1136 (1968).
36. M. Rosenblatt, *Stationary Sequences and Random Fields* (Birkhäuser, Boston, 1985).
37. W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.* **30**:134–144 (1989).
38. T. Subba Rao, On the theory of bilinear models, *J. Roy. Statist. Soc. Ser. B* **43**:244–255 (1981).
39. T. Subba Rao and M. M. Gabr, *An Introduction to Bispectral Analysis and Bilinear Time Series Models* (Springer, Berlin, 1984).
40. M. S. Taqqu, Law of the iterated logarithm for sums of non-linear functions of Gaussian that exhibit a long-range dependence, *Z. Wahrsch. Verw. Gebiete* **40**:203–238 (1977).
41. M. S. Taqqu, Convergence of integrated processes of arbitrary Hermite rank, *Z. Wahrsch. Verw. Gebiete* **50**:53–83 (1979).
42. G. Terdik, Bilinear Stochastic Models and related problems of nonlinear time series analysis, in *Lecture Notes in Statistics*, Vol. 142 (Springer-Verlag, New York, 1999).